Ma 341-002: Test II, systems of ODEs

For full credit make sure to show all work. If in doubt ask.

The "magic" formula may be useful for certain problems on this exam.

$$\boxed{e^{At} = e^{\lambda t}\left(I + t(A-\lambda I) + \frac{1}{2}t^2(A-\lambda I)^2 + \frac{1}{3!}t^3(A-\lambda I)^3 + \cdots\right)}$$

● **CHOOSE ONE OF THE PROOF PROBLEMS 1-5, IF YOU DO MORE THAN ONE I DO NOT INTEND TO REWARD WITH ANY BONUS POINTS. PLEASE JUST WORK ONE THAT YOU LIKE. THANKS.**

SOLUTION: see the practice test II from Fall 2007 for all of the proofs in one place. Of course these are also scattered about the notes here and there as well. Full credit was obtained for complete arguments devoid of incorrect statements.

(1.) [6pts] Show that $\vec{x} = e^{\lambda t}\vec{u}$ is a nonzero solution to $\vec{x}' = A\vec{x}$ if we require that λ and $\vec{u}$ are constant with $\det(A-\lambda I) = 0$ and $(A-\lambda I)\vec{u} = 0$. You will need to use a theorem from linear algebra which states that $B\vec{u} = 0$ has more than one solution only if $\det(B) = 0$.

(2.) [6pts] Show that if $\vec{w} = Re(\vec{w}) + iIm(\vec{w})$ is a solution to $\vec{w}' = A\vec{w}$ then both $Re(\vec{w})$ and $Im(\vec{w})$ are also solutions.

(3.) [6pts] Show that if $\lambda = \alpha + i\beta$ and $\vec{u} = \vec{a} + i\vec{b}$ then

$$\boxed{\begin{aligned} Re(e^{t}\vec{u}) &= e^{\alpha t}\cos(\beta t)\vec{a} - e^{\alpha t}\sin(\beta t)\vec{b} \\ Im(e^{\lambda t}\vec{u}) &= e^{\alpha t}\sin(\beta t)\vec{a} + e^{\alpha t}\cos(\beta t)\vec{b}. \end{aligned}}$$

Notice that we then have found how to extract two real solutions from the complex solution. I should mention that I assume here that $\alpha, \beta, \vec{a}, \vec{b}$ are all real, they have no $i = \sqrt{-1}$.

(4.) [6pts] Show that $\vec{x_p} = X\vec{v}$ is a solution to $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ if,

$$\boxed{\vec{x_p}(t) = X(t)\int X^{-1}(t)\vec{f}(t)dt.}$$

Here we assume X is a fundamental matrix for the system.

(5.) [6pts] Show that the matrix exponential is a fundamental matrix. That is show that e^{At} is invertible and it is a solution matrix for $\vec{x}' = A\vec{x}$.

PROBLEMS 6-10 ARE REQUIRED, YOU SHOULD ATTEMPT ALL OF THEM.

(6.) [30pts] Rewrite the following system of differential equations in matrix normal form

$$\boxed{x' = y \qquad\qquad y' = -5x - 4y.}$$

Then find the general solution using our eigenvalue/eigenvector technique. Finally, find the solution with $x(0) = 0$ and $y(0) = 1$ and write out the formulas for $x(t)$ and $y(t)$ separately.

SOLUTION: to begin we observe that the system is $\vec{x}' = A\vec{x}$ with $A = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix}$ so we can calculate the characteristic equation $\det(A - \lambda I) = 0$. Explicitly,

$$\det\begin{pmatrix} -\lambda & 1 \\ -5 & -\lambda - 4 \end{pmatrix} = \lambda(\lambda + 4) + 5 = \lambda^2 + 4\lambda + 5 = 0$$

Use the quadratic equation to see that our eigenvalues are $\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$. Let us chose $\lambda = -2 + i$ to avoid ambiguity in what follows. Find the eigenvector,

$$0 = (\ A - (-2+i)I\ )\vec{u} = \begin{pmatrix} 2-i & 1 \\ -5 & -2-i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

We could use either equation, I prefer the $(2-i)u + v = 0$ equation since it is easy to solve and obtain $v = (i-2)u$ thus if we chose $u = 1$ we find,

$$\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ i-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

I made the last step so that I could identify that $\vec{a} = [1, -2]^t$ and $\vec{b} = [0, 1]^t$ where the "t" stands for transpose, it simply makes the row vectors into column vectors. We find the general solution using the result of problem 3,

$$\vec{x} = c_1 e^{-2t}\left(\cos(t)\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + c_2 e^{-2t}\left(\sin(t)\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

we were given that $\vec{x}(0) = [0, 1]^t$ so we should apply this to specify c_1, c_2. Observe that since $\cos(0) = 1$, $e^{-2\cdot 0} = 1$ and $\sin(0) = 0$ our given initial condition says

$$\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ -2c_1 + c_2 \end{pmatrix} \quad \Longrightarrow \quad c_1 = 0,\ c_2 = 1.$$

Consequently, $\vec{x} = e^{-2t}\left(\sin(t)\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$. Therefore, we find that $x = e^{-2t}\sin(t)$ and $y = -2e^{-2t}\sin(t) + e^{-2t}\cos(t)$.

(7.) [30pts] Given that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & 6 \end{pmatrix}$$

find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$. You may need to use the matrix exponential and the magic formula to construct certain parts of the general solution.

SOLUTION: we begin by calculating the characteristic equation,

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -9 & 6-\lambda \end{pmatrix} \\ &= -\lambda \det \begin{pmatrix} -\lambda & 1 \\ -9 & 6-\lambda \end{pmatrix} - 1 \cdot 0 + 0 \cdot 0 \\ &= -\lambda(-\lambda(6-\lambda) + 9) \\ &= -\lambda(\lambda^2 - 6\lambda + 9) \\ &= -\lambda(\lambda - 3)^2 = 0 \end{aligned}$$

Clearly the eigenvalues for this problem are $\lambda_1 = 0, \lambda_2 = \lambda_3 = 3$. Our next step is to find the eigenvectors, begin with the zero eigenvalue. Look for $\vec{u}_1 = [u, v, w]^t$ such that

$$0 = (A - 0I)\vec{u}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & 6 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ -9v + 6w \end{pmatrix}.$$

the equations $v = 0, \ w = 0$ clearly are redundant with the third equation $-9v + 6w = 0$. What is missing ? An equation involving u, so u is a free variable and we choose $u = 1$. Consequently an eigenvector with eigenvalue zero is $\boxed{\vec{u}_1 = [1, 0, 0]^t}$.

Next, search for $\vec{u}_2 = [u, v, w]^t$ such that

$$0 = (A - 3I)\vec{u}_2 = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & -9 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -3u + v \\ -3v + w \\ -9v + 3w \end{pmatrix}$$

We observe that the third row equation is just the second row equation multiplied by 3. It is thus sufficient to focus our attention on the first two rows. This suggest that

$$\begin{aligned} v &= 3u \\ w &= 3w = 9u \end{aligned}$$

it is convenient to take the free variable to be u here, lets choose $u = 1$. $\boxed{\vec{u}_2 = [1, 3, 9]^t}$.

We have a system of three differential equations and three unknown dependent variables. The general theory told us that the general solution is comprised of three linearly independent solutions. Clearly we will only be able to build two eigenvector-type solutions. So we are missing a piece. We need to find a generalized eigenvector of order two corresponding to the repeated eigenvalue. We impose the chain condition $(A-3I)\vec{u}_3 = \vec{u}_2$ which automatically gives us that $(A-3I)^2\vec{u}_3 = 0$ since $\vec{u}_2$ is an eigenvector with eigenvalue 3.

$$(A-3I)\vec{u}_3 = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & -9 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -3u+v \\ -3v+w \\ -9v+3w \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

Consequently, $v = 3u+1,\ w = 3v+3$. We let $u = 0$ to obtain $v = 1,\ w = 3+3 = 6$. The choice $u = 0$ is allowed here because it does not result in forcing $\vec{u}_3 = 0$. We find that $\boxed{\vec{u}_3 = [0,1,6]^t\ .}$ There are many other choices which are equally valid. If you chose a different eigenvector for $\vec{u}_2$ that could make this part of the calculation look different from my solution as well, but again there are many correct solutions. They all return the same general solution. Our eigenvectors give us the two fundamental solutions,

$$\vec{x}_1 = e^{0\cdot t}\vec{u}_1 = \vec{u}_1 \qquad \textit{and} \qquad \vec{x}_2 = e^{3t}\vec{u}_2.$$

of course these can also be viewed as stemming from the matrix exponential if you prefer since $e^{At}\vec{u} = e^{\lambda t}\vec{u}$ for any eigenvector $\vec{u}$ with eigenvalue λ. However, what follows necessarily involves more than just eigenvectors. Since e^{At} is a fundamental matrix for the given system of ODEs we have that $e^{At}\vec{u}_3$ is a nontrivial solution of the system. Why? Because multiplication by $\vec{u}_3$ amounts to taking a linear combination of the columns of e^{At} which are (by problem 5) solutions. Our ODE is linear so the sum of solutions is again a solution. So there you have it, $\vec{x}_3 = e^{At}\vec{u}_3$ is a solution. Moreover, it is not one of the two we found already. This becomes evident once we calculate it explicitly,

$$\begin{aligned} \vec{x}_3 = e^{At}\vec{u}_3 &= e^{3t}(I\vec{u}_3 + t(A-3I)\vec{u}_3 + \frac{1}{2}t^2(A-3I)^2\vec{u}_3 + \cdots) \\ &= e^{3t}(\vec{u}_3 + t\vec{u}_2). \end{aligned}$$

Where I have used the "magic" formula and the fact that we constructed $\vec{u}_3$ to be a generalized eigenvector of order two satisfying the chain condition with $\vec{u}_2$. Thus the general solution is

$$\boxed{\vec{x} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + c_3 e^{3t}\left[\begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}\right]\ .}$$

(8.) [30pts] Solve $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ given that

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \vec{f} = \begin{pmatrix} 0 \\ 48e^{7t} \end{pmatrix}$$

SOLUTION: we calculate the characteristic equation and find our eigenvalues.

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0 \implies \lambda_1 = 1,\ \lambda_2 = -1.$$

Evidentally we will have no need for the matrix exponential in this problem because we have two distinct eigenvalues which will necessarily return two linearly independent eigenvectors. Lets find them.

$$0 = (A - I)\vec{u}_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u + v \\ u - v \end{pmatrix} \implies u = v \implies \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

I **chose** $u = 1$, it is not the only option, but it works. Next,

$$0 = (A + I)\vec{u}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + v \\ u + v \end{pmatrix} \implies v = -u \implies \vec{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Again there are other choices, but mine works fine. So we find the fundamental solutions to the homogeneous ODE $\vec{x}' = A\vec{x}$ are simply,

$$\vec{x}_1 = e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix} \qquad \text{and} \qquad \vec{x}_2 = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

Working towards finding $\vec{x}_p$, see problem 4. We form the fundamental matrix by concatenation of our solutions

$$X = [\vec{x}_1|\vec{x}_2] = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}$$

We can find the inverse of this matrix using the 2x2 formula for inverse,

$$X^{-1} = \frac{1}{-e^te^{-t} - e^te^{-t}}\begin{pmatrix} -e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-t} & e^{-t} \\ e^t & -e^t \end{pmatrix}$$

Its easy to check our work to see that $XX^{-1} = I$.

Now we find the particular solution using the result of problem 4.

$$\begin{aligned}
\vec{x}_p &= X\int X^{-1}\vec{f}dt \\
&= X\int \frac{1}{2}\begin{pmatrix} e^{-t} & e^{-t} \\ e^{t} & -e^{t}\end{pmatrix}\begin{pmatrix} 0 \\ 48e^{7t}\end{pmatrix} dt \\
&= X\int \begin{pmatrix} 24e^{6t} \\ -24e^{8t}\end{pmatrix} dt \\
&= X\begin{pmatrix} \int 24e^{6t}dt \\ \int -24e^{8t}dt\end{pmatrix} \\
&= \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t}\end{pmatrix}\begin{pmatrix} 4e^{6t} \\ -3e^{8t}\end{pmatrix} \\
&= \begin{pmatrix} e^{7t} \\ 7e^{7t}\end{pmatrix}
\end{aligned}$$

Therefore, the general solution is

$$\boxed{\vec{x} = c_1e^{t}\begin{pmatrix} 1 \\ 1\end{pmatrix} + c_2e^{-t}\begin{pmatrix} 1 \\ -1\end{pmatrix} + \begin{pmatrix} e^{7t} \\ 7e^{7t}\end{pmatrix}.}$$

(9.) [5pts] Suppose that we have a system of differential equations in matrix form $\frac{d\vec{x}}{dt} = A\vec{x}$ for some constant matrix A. In addition suppose that this system has solutions $\vec{x}_1 = te^{t}\vec{u}_1$ and $\vec{x}_2 = \cos(t)\vec{u}_2$. What is the minimum size for such a system ? That is how big is the matrix at a minimum ?

<u>SOLUTION:</u> First notice that the only way we can get a $te^{t}\vec{u}_1$ is to have the eigenvalue $\lambda = 1$ repeated at least once. Remember I mentioned several times that there always exists at least one eigenvector for each distinct eigenvalue. So in order for us to find $te^{t}\vec{u}_1$ as a solution (which comes from a generalized eigenvector) there must also be $e^{t}\vec{u}$-type solution. Clearly these are linearly independent solutions, they are not constant multiples of one another.

Second, in order for us to have $\vec{x}_2 = \cos(t)\vec{u}_2$ as a solution there must exist another solution of the form $\sin(t)\vec{u}_2$ because the solutions from complex eigenvalues always come in pairs (see problem 3 of this test). Here the eigenvalue was $\lambda = \pm i$. It is an inescapable fact of algebra that complex roots of real polynomials always come in complex conjugate pairs. (otherwise when you foiled out the roots you'd find the real polynomial was not real)

We could add more eigenvalues for some system with these solutions but we cannot take them away, ***the smallest system would be a 4x4***. For example, you could convert the 4$^{\text{th}}$ order ODE $(D-1)^2(D^2+1)[y] = 0$ into a matrix ODE in normal form and it would possess solutions such as those given at the beginning of the problem. We must have a 4x4 system in order to get 4 linearly independent solutions.

(10.) [4pts] There are many systems of differential equations which correspond to a single higher order constant coefficient differential equation. For example, I constructed problem 7 of this test from the 3rd order ODE $y''' - 6y'' + 9y' = 0$. Is there a corresponding 2nd order ODE for a 2x2 system $\frac{d\vec{x}}{dt} = A\vec{x}$ with solutions

$$\vec{x}_1 = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{x}_2 = e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

if so find that 2nd order ODE, if not explain why not.

SOLUTION: In short, no. The reason is as follows. If there were such a 2nd order ODE it would have the eigenvalue $\lambda = 1$ repeated twice. We know this because a system of n ODEs and the constant coefficient n-th order ODE to which it corresponds share the same algebraic characterization. I mean to say that the auxillarly equation and the characteristic equation are identical (see homework problem §9.5#41 solution on H87, the proof actually generalizes to cases with $n > 2$ but we only needed $n = 2$ for this question). Whether we view the problem as an n-th order ODE or a system of n first order ODEs we face the same algebra problem. So getting back to the question at hand we have the eigenvalue $\lambda = 1$ repeated twice. We learned in the first third of this course that our solution would have the form $y = c_1e^t + c_2te^t$. I argue that the appearance of the te^t-term ruins our hopes of matching our given system $\frac{d\vec{x}}{dt} = A\vec{x}$. For the given system there are two eigenvectors; $\vec{u}_1 = [1, 0]^t$ and $\vec{u}_2 = [0, 1]^t$ and we already have by inspection two linearly independent solutions. For a 2x2 solution that is all we can find. There is no room for three linearly independent solutions in a two-dimensional problem. But to make the correspondence that is just what we would need, we need to have hope that our system had a generalized eigenvector that could generate te^t-type terms.

If I had only given $\vec{x}_1$ or $\vec{x}_2$ but not both then you could have constructed such a system. In fact, E3 on page 113 of my notes is just such a problem.

As I write this solution I'm sure many of you think you have constructed such a system, but I'd wager a closer inspection of your solution will reveal that $\vec{x}_1, \vec{x}_2$ are solutions for the A you calculated BUT that system does not in fact correspond to any 2nd order constant coefficient ODE. You see the point of this problem was to explore why the class of matrix problems correspondent to n-th order ODEs is smaller than the general class of matrix problems. This is similar to the phase plane discussion where the phase plane problems stemming from Newton's Law have very special properties that the more abstract phase plane problems do not generally possess.